so p is certainly null-homologous, however, this element does not equal 1 in $\pi_1(\mathscr{H})$, since $\pi_1(\mathscr{H})$ is the free group generated by a, b.

EXERCISE 5.1.4.1. Show that $waba^{-1}b^{-1}w^{-1}$ is freely equivalent to a product of commutators, so that the commutator subgroup of a group G is in fact generated by the commutators of G.

Give an example to show that the commutators of generators of G do not in general suffice to generate its commutator subgroup.

EXERCISE 5.1.4.2. If p is the boundary of a singular perforated orientable surface in $\mathscr{C}$ prove that $[p]$ is in the commutator subgroup of $\pi_1(\mathscr{C})$.

5.2 The Structure Theorem for Finitely Generated Abelian Groups

5.2.1 Introduction

The fundamental theorem for finite abelian groups appears in Kronecker 1870. In this paper, Kronecker gives what we would recognize as the abstract definition of a finite abelian group—a finite set closed under a commutative, associative binary operation f, with the property that $a' \neq a''$ implies $f(a, a') \neq f(a, a'')$—then proves that such a group is a direct product of cyclic groups. Kronecker's proof is so brief and lucid we shall reproduce it almost verbatim below.

A different proof, using matrices, was discovered by Poincaré 1900. Poincaré's method is actually intended to compute the Betti number and torsion coefficients (of given dimension) of a complex, but this is tantamount to decomposing a finitely generated abelian group into certain cyclic factors, the number of infinite cyclic factors being the Betti number, and the orders of the finite factors being the torsion coefficients. His result is therefore a generalization of Kronecker's—what we now know as the structure theorem for finitely generated abelian groups—however, we shall see how Kronecker's proof can be augmented to deal with elements of infinite order. (This seems to have first been done by Noether 1926.)

Kronecker's proof begins with the following remarks.

(1) The exponents k of all powers a^k equal to 1 for a fixed element a are integer multiples of some positive integer n called the *period* of a.
(2) If n is a period, so is any divisor of n.
(3) If a', a'' have periods n', n'' which are relatively prime, then $a'a''$ has period $n'n''$.
(4) If n_1 is the lowest common multiple of the periods of elements in the group, then there is in fact an element of period n_1. For if

$$n_1 = p^\alpha q^\beta r^\gamma \cdots$$

is the prime factorization of n_1, there must be periods n containing $p^\alpha, q^\beta, r^\gamma, \ldots$ as factors, and hence by (2), elements $a', a'', a''', \ldots$ of periods $p^\alpha, q^\beta, r^\gamma, \ldots$ respectively. Then by (3) the element $a'a''a''' \cdots$ has period $p^\alpha q^\beta r^\gamma \cdots = n_1$.

It will be seen from the proof which follows that Kronecker is implicitly using coset decompositions and coset representatives, however, the directness of his argument is more obvious if these terms are not mentioned.

5.2.2 Kronecker's Theorem

If A is a finite abelian group, then $A = A_1 \times A_2 \times \cdots \times A_s$, where $A_1, A_2, \ldots$ are cyclic groups of orders $n_1, n_2, \ldots$ and each n_{i+1} is a divisor of n_i.

Let n_1 denote, as in (4), the maximal period among elements of A. Then n_1 is a multiple of the period of each element a, and we have

$$a^{n_1} = 1$$

for an arbitrary $a \in A$.

If a_1 is an element with period n_1, we shall call elements a', a'' *equivalent relative* to a_1 if

$$a'a_1^k = a'' \quad \text{for some } k.$$

This is indeed an equivalence relation, and the equivalence classes form a finite abelian group under the obvious multiplication (it is, of course, the quotient of A by the cyclic subgroup generated by a_1). The properties (1)–(4) relativize to corresponding properties of equivalence. In particular, there is an equivalence class of maximal period n_2, which means that for any representative a^* of the class, $(a^*)^{n_2}$ is the least of its powers equivalent to 1. Since $(a^*)^{n_1}$ equals 1 and is *a fortiori* equivalent to it, the relativized version of (1) says that n_2 is a divisor of n_1.

Now if $(a^*)^{n_2} = a_1^k$ and one raises both sides to the power n_1/n_2 then

$$1 = (a^*)^{n_1} = a_1^{kn_1/n_2}$$

so when k/n_2 is set equal to m we have

$$a_1^{mn_1} = 1$$

from which it follows, since n_1 is the period of a_1, that m is an integer.

The equation

$$a_2 a_1^m = a^*$$

then defines an element a_2 equivalent to a^* whose n_2th power is not merely equivalent to 1, but equal to it.

We now call elements a', a'' *equivalent relative to* a_1, a_2 if

$$a'a_1^h a_2^k = a'' \quad \text{for some } h, k$$

and similarly obtain a group of equivalence classes whose maximal period, n_3, divides n_2, and a representative a_3 of the class of maximal period such that $a_3^{n_3} = 1$.

The procedure terminates when we have a set of elements $a_1, a_2, \ldots, a_s$ such that any a is equivalent to 1 relative to $a_1, a_2, \ldots, a_s$, that is, when any a is expressible as

$$a = a_1^{h_1} a_2^{h_2} \cdots a_s^{h_s} \qquad (0 \leq h_i < n_i).$$

It also follows that the expression is unique, for the equivalence classes relative to $a_1, \ldots, a_{s-1}$ must constitute a cyclic group with a_s as a representative generator. An element a is therefore uniquely determined by the integers $h_1, \ldots, h_{s-1}$, which determine it relative to an equivalence class representative, and the integer h_s which determines the equivalence class representative itself, $a_s^{h_s}$.

Thus A is the direct product $A_1 \times A_2 \times \cdots \times A_s$, where A_i is the cyclic group generated by a_i, and the order n_i of A_i is such that n_{i+1} divides n_i.

□

Note that the orders $n_1, n_2, \ldots, n_s$ of factors in a representation of A as a direct product of cyclic groups are uniquely determined by the requirement that n_{i+1} divides n_i. For in a group $A_1 \times A_2 \times \cdots \times A_s$ in which the orders n_i of A_i have this property, n_1 is indeed the maximal period of an element, n_2 the maximal period in the quotient modulo the subgroup generated by an element of order n_1, etc.

One can actually factor further into cyclic subgroups whose orders are prime powers—for example, the cyclic group of order 6 is the direct product of cyclic groups of orders 2 and 3—however, the numbers $n_1, n_2, \ldots, n_s$ are those most suitable to describe the "torsion" in the group. If A is the first homology group of a complex, n_1 represents the maximum number of times a nonbounding curve has to be traversed before it becomes bounding, n_2 is the maximum when curves are considered "relative to a curve of period n_1," and so on.

5.2.3 A Factorization Theorem

If A is an abelian group and B a subgroup such that A/B is free abelian, then

$$A = B \times \frac{A}{B}$$

(Note: in what follows we understand "free generators," "nontrivial relation," and so on in the context of abelian groups. For example, $a_1 a_2 a_1^{-1} a_2^{-1} = 1$ is now a *trivial* relation.)

Let $x_1, x_2, \ldots$ be free generators of A/B and for each i choose a $c_i \in A$ such that $\phi(c_i) = x_i$, where $\phi: A \to A/B$ is the canonical homomorphism. Then the c_i freely generate a subgroup C of A, isomorphic to A/B, since any nontrivial relation between the c_i's would yield the corresponding relation between the x_i's under the map ϕ.

It follows that any $a \in A$ has a unique factorization $a = bc$, where $b \in B$, $c \in C$. For c must satisfy $\phi(c) = \phi(a)$, and there is exactly one such c by the construction and freeness of C; and b is then uniquely determined as ac^{-1}. The latter is indeed an element of B, since $\phi(ac^{-1}) = \phi(a)(\phi(c))^{-1}$.

Now if $a_1 = b_1c_1, a_2 = b_2c_2$ we have $a_1a_2 = (b_1b_2)(c_1c_2)$, so multiplication takes place componentwise on the B and C factors. In other words, $A = B \times C$ or $B \times A/B$, since C is isomorphic to A/B. □

5.2.4 Free Abelian Groups of Finite Rank

If $\mathbf{Z}^n$ denotes the abelian group freely generated by $a_1, \ldots, a_n$, then any set of free generators for $\mathbf{Z}^n$ has n elements. Also, any subgroup of $\mathbf{Z}^n$ is free abelian, with $\leq n$ generators.

The typical element $a = a_1^{k_1}a_2^{k_2} \cdots a_n^{k_n}$ of $\mathbf{Z}^n$ can be represented by the integer vector $\boldsymbol{a} = (k_1, k_2, \ldots, k_n)$ and the product operation in $\mathbf{Z}^n$ then corresponds to vector sum.

By elementary linear algebra, a set of $>n$ such vectors is linearly dependent with rational, and hence in fact integer, coefficients (multiplying through by a common denominator). This means there is a nontrivial relation between any set of $>n$ members of $\mathbf{Z}^n$; so any set of free generators has $\leq n$ members. Conversely, if $b_1, \ldots, b_m$ generate $\mathbf{Z}^n$, then the elements $a_1, \ldots, a_n$ in particular must be products of them. In other words, the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linear combinations of the $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$. Since $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linearly independent, $m \geq n$ by the same argument. Hence $m = n$.

The number n is the number of factors in the decomposition of $\mathbf{Z}^n$ into the direct product of infinite cyclic groups, so we have now shown that this decomposition is unique.

To show the second part of the theorem, suppose that C is a subgroup of $\mathbf{Z}^n$. We observe that a subgroup of $\mathbf{Z}^1$ is certainly free, on $\leq$ one generator, and continue by induction on n as follows.

The projection $\pi: \mathbf{Z}^n \to \mathbf{Z}^1$ which sends each a_i to a_1 maps C onto a free subgroup C_1 of $\mathbf{Z}^1$. Then by the factorization theorem

$$C = B_1 \times C_1,$$

where B_1 is the kernel of the projection $C \to C_1$, that is, a subgroup of the free abelian group generated by $a_2, \ldots, a_n$. By induction we can assume that B_1 is free abelian on $\leq n - 1$ generators, that is, the direct product of $\leq n - 1$

infinite cyclic groups, and then C is the direct product of $\leq n$ infinite cyclic groups. □

EXERCISE 5.2.4.1. Prove that finitely generated abelian groups are finitely presented. (Of course, this result will follow from the structure theorem, but it is of interest to see what it really depends on.)

5.2.5 Torsion-free Abelian Groups

An abelian group A is called torsion-free if it has no elements of finite order. A finitely-generated torsion-free abelian group is free.

Let $a_1, \dots, a_n$ be a maximal subset of the generators of A which generate freely. Then for each $i > n$ the generator a_i enters a nontrivial relation with $a_1, \dots, a_n$, which we may assume to be

$$w(a_1, \dots, a_n) = a_i^{k_i}.$$

Thus if B denotes the free abelian group generated by $a_1, \dots, a_n$ we have $a_i^{k_i} \in B$ for each $i > n$. Let k be a common multiple of the k_i's and consider the homomorphism $\phi: A \to B$ which sends each a_i to a_i^k. Since no a_i has finite order the kernel is trivial and hence we have a monomorphism. The image subgroup of B is free by 5.2.4, so A itself is free. □

We mention in passing that an infinitely-generated torsion-free abelian group may not be free—an interesting example is the group D of rationals of the form $p/2^q$ (p, q integers) under addition. Exercise 5.2.5.1 develops some of the properties of this group, which actually occurs in topology (see Rolfsen 1976, p. 186).

For the theory of infinitely-generated abelian groups, which is quite well developed, see Fuchs 1960.

EXERCISE 5.2.5.1. (1) Show that any finite set of elements $p_1/2^{q_1}, \dots, p_n/2^{q_n}$ of D with $q_1 \leq \dots \leq q_n$ generate an infinite cyclic subgroup containing no element $< 1/2^{q_n}$. Deduce that D is not finitely generated and that any finite set of ≥ 2 elements satisfy a nontrivial relation.

(2) Show that D has a presentation

$$\langle a_1, a_2, a_3, \dots; a_1 = a_2^2, a_2 = a_3^2, a_3 = a_4^2, \dots \rangle.$$

(3) Show that every proper subgroup $\neq \{1\}$ and containing the element a_1 of D is infinite cyclic.

(4) Show that $D/\mathbf{Z}$, the result of adding the relation $a_1 = 1$ to D, is an infinite group whose proper subgroups are all finite cyclic.

EXERCISE 5.2.5.2. Show that the positive rationals under multiplication constitute an infinitely generated free abelian group.

5.2.6 The Torsion Subgroup

Suppose A is any finitely-generated abelian group, and let T be the subgroup of elements of finite order. T is called the torsion subgroup. *Then T is a finite abelian group and $A = F \times T$ where F is a free abelian group.*

Let $\mathbf{Z}^k$ be the free abelian group on the generators of A and let B be the subgroup of $\mathbf{Z}^k$ which maps onto T under the canonical homomorphism $\phi\colon \mathbf{Z}^k \to A$. By 5.2.4, B is finitely generated, so the images of its generators give a finite set of generators for T. But a finitely generated abelian group in which every element has finite order is obviously finite, hence T is a finite abelian group.

Now consider the coset decomposition of A modulo T. If any coset is of order m this means $x^m \in T$ for any of its representatives x. But then x^m is of finite order, hence so is x and the coset in question can only be T itself. Thus A/T is torsion-free and hence free by 5.2.5.

It then follows by the factorization theorem that $A = F \times T$ where F is the free abelian group A/T. □

The proof of the structure theorem is now complete. We have decomposed the given finitely generated abelian group A into the direct product of a free abelian group A/T and a finite abelian group T. This decomposition is unique because in any abelian group $F \times T$, where F is free and T is finite the torsion subgroup is obviously T. The free abelian group A/T decomposes uniquely into the direct product of n infinite cyclic groups by 5.2.4, while T decomposes uniquely into cyclic groups of orders $n_1, \ldots, n_s$, where n_{i+1} divides n_i, by Kronecker's theorem and the remark following it in 5.2.2. A is therefore uniquely determined by the number n (Betti number) and the numbers $n_1, \ldots, n_s$ (torsion coefficients).

5.2.7 Computability of the Betti Number and Torsion Coefficients

The above proof of the structure theorem does not make clear how to actually compute the decomposition of a given finitely-generated abelian group A into cyclic factors. The proof using matrices is quite explicit in this respect (see for example Cairns 1961), however, we can also obtain an algorithm for computing the decomposition from its mere existence by the following cheap trick:

Given a presentation of A, systematically apply all possible Tietze transformations until an abelian presentation of the form

$$\langle a_1, \ldots, a_n, b_1, \ldots, b_s;\, b_1^{n_1}, \ldots, b_s^{n_s}\rangle,$$

where each n_{i+1} divides n_i, is obtained. Then n is the Betti number of A and $n_1, \ldots, n_s$ are its torsion coefficients. The structure theorem implies the existence of such a presentation, so we must be able to reach it in a finite